Isoperimetric regions in $\mathbb{H}^2$ between parallel horocycles

Márcio Fabiano da Silva

Abstract. In this work we investigate the following isoperimetric problem in the hyperbolic plane: to find the regions of prescribed area with minimal perimeter between two parallel horocycles. We give an explicit and detailed description of all such regions.

1. Introduction

For a Riemannian manifold M, the classical isoperimetric problem consists in classifying, up to congruency by the isometry group of M, the (compact) regions $\Omega \subseteq M$ enclosing a fixed volume that have minimal boundary volume. The existence and regularity of solutions for a large number of cases may be guaranteed by adapting some results from the Geometric Measure Theory (cf. [6]).

For example, when M is the Euclidean plane $\mathbb{R}^2$, the classical isoperimetric problem has the disk as the unique solution. If M is a hyperbolic surface, the least-perimeter enclosures of prescribed area are described in [1] and [7]. An interesting version of the isoperimetric problem is to study it in a slab. Physically, it corresponds to determine the shape of a drop trapped between two parallel planes, which was solved by Vogel in [8]. Independently, Athanassenas studied the isoperimetric problem between parallel planes of

$\mathbb{R}^3$ in [2]. If M is a slab between two parallel horospheres in the 3-dimensional hyperbolic space $\mathbb{H}^3(-1)$, the possible isoperimetric regions were obtained in [3].

In this paper we will use the upper halfplane model $\mathbb{R}^2_+$. The parallel horocycles are represented by the horizontal straight lines of $\mathbb{R}^2_+$. We will present in this paper a detailed and complete classification of the isoperimetric solutions.

In Section 2 we give some basic definitions in the model $\mathbb{R}^2_+$ for the hyperbolic plane like geodesics and curves of constant geodesic curvature. We also present a more precise formulation for the considered isoperimetric problem and get some preliminary characterizations by adapting the results from [3]. We will see that the possible isoperimetric regions must be delimited by such curves and meet the horocycles perpendicularly when this intersection is nonempty. We introduce the so-called geodesic halfdisk, horocycle halfdisk and equidistant halfdisk.

In Section 3 we read off the expressions for perimeter and area for the regions obtained in Section 2 as the possible isoperimetric solutions.

In Section 4 we compare the perimeter of the possible isoperimetric regions with prescribed area. In fact, we see that this is equivalent to investigating the regions of maximal area with prescribed perimeter.

In Section 5 we give the isoperimetric profile for the region between two parallel horocycles in $\mathbb{R}^2_+$ and prove the following result:

Let c be a positive real constant and $\mathcal{F}_c = \{(x,y) \in \mathbb{R}^2_+ : 1 \leq y \leq c\}$. Let $A > 0$ and $\mathcal{C}_{c,A}$ be the set of all $\Omega \subset \mathcal{F}_c$ with area $|\Omega| = A$ and perimeter $|\partial(\Omega \cap \mathring{\mathcal{F}}_c)| < \infty$, where we suppose Ω to be connected, compact, 2-rectifiable

in $\mathcal{F}_c$, having as boundary (between the horocycles) a simple rectifiable curve.

Theorem 1.1. *Let* $L_{c,A} = \inf\{|\partial(\Omega \cap \mathring{\mathcal{F}}_c)| : \Omega \in \mathcal{C}_{c,A}\}$. *Then*

1. *there exists* $\Omega \in \mathcal{C}_{c,A}$ *such that* $|\partial(\Omega \cap \mathring{\mathcal{F}}_c)| = L_{c,A}$;

2. *if* $\Omega \subset \mathcal{F}_c$ *has minimal perimeter, the boundary of* Ω *has a single connected component made up with either*

   (a) *a halfdisk (geodesic, horocycle, equidistant) above* $\{y = 1\}$;

   (b) *a section of* $\mathcal{F}_c$, *namely*

$$S_{[x_0,x_1]} = [x_0, x_1] \times [1, c].$$

More precisely, if d *is the hyperbolic distance between the horocycles, we have:*

1. *if* $d < 1$, *there exists* $A_0(c)$ *such that*

- *if* $A < A_0(c)$ *then* Ω *is a geodesic halfdisk;*
- *if* $A = A_0(c)$ *then* Ω *is a geodesic halfdisk or a section;*
- *if* $A > A_0(c)$ *then* Ω *is a section;*

2. *if* $d = 1$, *there exists* $A_0(c)$ *such that*

- *if* $A < A_0(c)$ *then* Ω *is a geodesic halfdisk;*
- *if* $A = A_0(c)$ *then* Ω *is a horocycle halfdisk or a section;*
- *if* $A > A_0(c)$ *then* Ω *is a section;*

3. *if* $d > 1$, *there exist two constants* $A_0(c) < A_1(c)$ *such that*

- *if* $A < A_0(c)$ *then* Ω *is a geodesic halfdisk;*

- *if $A = A_0(c)$ then Ω is a horocycle halfdisk;*
- *if $A_0(c) < A < A_1(c)$ then Ω is an equidistant halfdisk;*
- *if $A = A_1(c)$ then Ω is an equidistant halfdisk or a section;*
- *if $A > A_1(c)$ then Ω is a section.*

Acknowledgments. The present work was supported by CAPES and CNPq. The author is specially grateful to Professor Doctor Luiz Amancio Machado de Sousa Junior, from Universidade do Rio de Janeiro, for his helpful suggestions.

2. Preliminaries

In this section we will introduce some basic facts and notations that will be used along the paper. There is a large literature about the subject (we suggest beginning with [4]). We also adapt some important results of [3] to get the possible isoperimetric regions in the hyperbolic plane.

Let $\mathcal{L}^3 = (\mathbb{R}^3, g)$ be the 3-dimensional Lorentz space endowed with the metric $g(x, y) = x_1y_1 + x_2y_2 - x_3y_3$ and the hyperbolic plane

$$\mathbb{H}^2 := \{p = (x_1, x_2, x_3) \in \mathcal{L}^3 : \ g(p, p) = -1, \ x_3 > 0\}.$$

We use the upper halfplane model $\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2; y > 0\}$ for $\mathbb{H}^2$, endowed with the metric $<,>= ds^2 = \dfrac{dx^2 + dy^2}{y^2}$.

The Euclidean straight line $\{y = 0\}$ is the infinity boundary of $\mathbb{R}^2_+$, denoted by $\partial_\infty \mathbb{R}^2_+$.

The curves of constant geodesic curvature $k \geq 0$ in $\mathbb{R}^2_+$ are described as follows:

1. *Geodesic:* $(k = 0)$. Represented by vertical Euclidean straight lines contained in $\mathbb{R}^2_+$ and Euclidean semicircles perpendicular to $\partial_\infty \mathbb{R}^2_+$ and contained in $\mathbb{R}^2_+$;

2. *Geodesic circles:* $(\ k > 1)$. Represented by Euclidean circles entirely contained in $\mathbb{R}^2_+$;

3. *Horocycles:* $(k = 1)$. Represented by horizontal Euclidean straight lines of $\mathbb{R}^2_+$ and Euclidean circles of $\mathbb{R}^2_+$ tangent to $\partial_\infty \mathbb{R}^2_+$.

4. *Equidistant curves:* $(0 < k < 1)$. Represented by the intersection of $\mathbb{R}^2_+$ with the straight lines of $\mathbb{R}^2$ that are neither parallel nor perpendicular to $\{y = 0\}$, and by the Euclidean circles not entirely contained in $\mathbb{R}^2_+$ and are neither tangent nor perpendicular to $\{y = 0\}$.

The isometries of $\mathbb{R}^2_+$ are the Möbius transformations of $\widehat{\mathbb{C}}$ that leave $\mathbb{R}^2_+$ invariant. For our purposes, we are interested in the following Euclidean applications: horizontal translations, reflections with respect to a vertical geodesic, homotheties and inversions (with respect to circles centered in $\{y = 0\}$).

For $\mathbb{R}^2_+$ the isoperimetric problem may be formulated as follows: "to minimize the perimeter of a region inside two parallel horocycles (represented by two horizontal Euclidean straight lines), with prescribed area, but not counting its part of the boundary contained in the horocycles". By the *perimeter* of a region we mean the length of its boundary.

Since the Euclidean homothety is an isometry of $\mathbb{R}^2_+$, we take the lower horocycle as $\{y = 1\}$ to study the isoperimetric problem, so that any solution is obtained by homothety.

By adapting the demonstration of Theorem 1.1 from [3] to our case, namely $\mathbb{R}^2_+$, together with Lemma 2.1 of [1], we have that there exists regular isoperimetric solutions and they are regions whose boundary consists of curves of constant geodesic curvature perpendicular to the horocycles (when the intersection is non-empty). Essentially, this proves the first item of our Theorem 1.1 in this present paper, stated at the Introduction.

Before we start to calculate the expressions for the perimeter and area of the regions delimited by curves of constant geodesic curvature, we present the polar coordinate system for $\mathbb{R}^2_+$ and conclude this section by giving a more precise formulation for the isoperimetric problem.

If (x, y) are the cartesian coordinates in $\mathbb{R}^2_+$ and γ is the geodesic $y > 0$, we define the polar coordinates (ρ, θ) of a point $p \in \mathbb{R}^2_+$ as follows: ρ is the hyperbolic distance from p to the origin $O = (0, 1)$ and θ is the angle between a fixed geodesic radius γ^+, given by $\{x = 0;\ y \geq 1\}$, and the geodesic through O and p, measured counterclockwise.

The relation between these systems of coordinates is:

$$(x, y) = \frac{1}{\cosh\rho - \sinh\rho\cos\theta}\,(\sinh\rho\sin\theta, 1), \tag{1}$$

and the metric of $\mathbb{R}^2_+$ in polar coordinates is $d\sigma^2 = d\rho^2 + \sinh^2\rho\ d\theta^2$.

We now obtain the expression for the arclength of a geodesic circle and the area of a sector as functions of the central angle β. For the sake of simplicity we take the circle of hyperbolic radius ρ centered in O.

The geodesic circle can be parametrized by $\alpha(\theta) = (\rho, \theta)$, with constant ρ and $0 \leq \theta \leq \beta$. Then $d\sigma^2(\alpha') = \sinh^2\rho$. Therefore, the arclength corre-

sponding to β in the hyperbolic metric is

$$L(\alpha) = \int_0^\beta \sqrt{d\sigma^2(\alpha')}\ d\theta = \beta\ \sinh\rho, \tag{2}$$

and the area A of a sector of the disk corresponding to β is

$$A = \int_0^\beta \int_0^\rho \sinh\rho\ d\rho\ d\theta = \beta\ (\cosh\rho - 1). \tag{3}$$

As we mentioned above, the isoperimetric solutions are regions delimited by curves of constant geodesic curvature perpendicular to the horocycles (when the intersection is non-empty). So we have the following possibilities for barriers: vertical geodesics, geodesic circles, horocycles represented by Euclidean circles of $\mathbb{R}^2_+$ tangent to $\partial_\infty \mathbb{R}^2_+$, and equidistant curves represented by Euclidean circles not entirely contained in $\mathbb{R}^2_+$ and neither tangent nor perpendicular to $\{y = 0\}$. The region in $\mathcal{F}_c$ delimited two vertical geodesics will be called a *section*. The region in $\mathcal{F}_c$ delimited by geodesic circles perpendicular to $\{y = 1\}$ or $\{y = c\}$ will be called *geodesic halfdisk*. The region in $\mathcal{F}_c$ delimited by horocycles and equidistant curves perpendicular to $\{y = 1\}$ will be called *horocycle halfdisk* and *equidistant halfdisk*, respectively. We mean by *halfdisk above* (respectively *below*) $\{y = c\}$, the part of the Euclidean halfdisk above (respectively below) the horocycle $\{y = c\}$ (see Figure 4).

Isoperimetric problem for $\mathcal{F}_c$: *fix an area value and study the domains* $\Omega \subset \mathcal{F}_c$ *with the prescribed area which have minimal free boundary perimeter.*

Definition 2.1: A (compact) minimizing region Ω for this problem will be called an *isoperimetric solution* or *region* in $\mathcal{F}_c$.

3. Expression for perimeter and area

In this section we get expressions for the perimeter and area of the possible isoperimetric solutions Ω contained in $\mathcal{F}_c$. For our purposes we consider only the regions that are 2(-dimensional)-rectifiable (with respect to Hausdorff's measure) with boundary 1(-dimensional)-rectifiable. We denote this measure by $|\cdot|$, so that any Ω has area $|\Omega|$ and perimeter $|\partial\Omega|$, but it *never* counts $\partial\Omega \cap \partial\mathcal{F}_c$. For more details, see [6].

3.1. Perimeter and area of a section

Let $c > 1$ and $x_0 < x_1$ be real constants. For the sake of simplicity, we consider the vertical geodesics $\{x = x_0\}$ and $\{x = x_1\}$, contained in $\mathbb{R}^2_+$, and the parallel horocycles $\{y = 1\}$ and $\{y = c\}$.

Lemma 3.1.1. *Under the notations above, if T is a section then*

$$|\partial T| = 2 \ln c \quad \textit{and} \quad |T| = (x_1 - x_0)(-1/c + 1).$$

Proof: Since the length of a vertical geodesic segment $1 < y < c$ is $\ln(c/1) = \ln c$, then $|\partial T| = 2 \ln c$. And

$$|T| = \int_{x_0}^{x_1} \int_1^c \frac{1}{y^2} \, dy \, dx = (x_1 - x_0)(-1/c + 1).$$

q.e.d.

3.2. Perimeter and area for a geodesic halfdisk and a horocycle halfdisk

Consider $c \in \mathbb{R}_+^*$ and $\{y = c\}$ a horocycle in $\mathbb{R}_+^2$. We take the Euclidean circle S centered in $(0, c)$ with radius $r < c$. The circle S can be viewed as a geodesic circle S_H with hyperbolic center $C_H = (0, h)$ and hyperbolic radius ρ. We want to relate the centers and the radii of S and S_H.

In the hyperbolic metric, since C_H equidists from both $(0, c - r)$ and $(0, c + r)$, it is easy to see that $h = \sqrt{c^2 - r^2}$, $C_H = (0, \sqrt{c^2 - r^2})$ and

$$\rho = \int_{c-r}^{h} \frac{1}{t}\, dt = \ln \frac{h}{c - r} = \frac{1}{2}\, \ln\left(\frac{c + r}{c - r}\right). \tag{4}$$

From (4) we have that $\dfrac{r}{c} = \dfrac{e^{2\rho} - 1}{e^{2\rho} + 1} = \tanh \rho$.

Later we will use the relation between $|S^+|$ and $|S^-|$, where S^+ and S^- are halfdisks above and below $\{y = c\}$, respectively. They are given by

$$|S^+| = 2 \int_0^r \int_c^{c+\sqrt{r^2-x^2}} \frac{1}{y^2}\, dy\, dx = \frac{2}{c} \int_0^r \frac{\sqrt{r^2 - x^2}}{c + \sqrt{r^2 - x^2}}\, dx,$$

and

$$|S^-| = 2 \int_0^r \int_{c-\sqrt{r^2-x^2}}^{c} \frac{1}{y^2}\, dy\, dx = \frac{2}{c} \int_0^r \frac{\sqrt{r^2 - x^2}}{c - \sqrt{r^2 - x^2}}\, dx.$$

Notice that $|S^-| > |S^+|$. Similarly, one has

$$|\partial S^-| > |\partial S^+|. \tag{5}$$

In Figure 1, θ denotes the angle between $\{x = 0, y \geq h\}$ and the geodesic $\tilde{S}$ through C_H and (r, c). Thus, θ measures the half of the central angle corresponding to the arc of the geodesic semicircle above $\{y = c\}$, and $\tilde{S}$ has center $(r, 0)$ and radius c.

Since the Euclidean and hyperbolic metrics are conformal, in order to measure θ we parametrize $\tilde{S}$ as $\alpha(t) = (c\sin t + r, c\cos t)$, with $-\pi/2 < t < \pi/2$. Then $C_H = (0, \sqrt{c^2 - r^2}) = \alpha(t_0) = (c\sin t_0 + r, c\cos t_0)$ so that $\sin t_0 = -r/c$ and

$$\cos\theta = -\sin t_0 = r/c. \tag{6}$$

If $\bar{S}$ is the region delimited by $\tilde{S}$, axis y and $\{y = c\}$ then

$$|\bar{S}| = \int_h^c \frac{r - \sqrt{c^2 - y^2}}{y^2}\, dy = -r/c + \pi/2 - \arcsin(h/c).$$

Suppose $c > 1$ and consider the parallel horocycles $\{y = 1\}$ and $\{y = c\}$. Let S_1 be the circle centered at $(0, 1)$ with radius $r_1 < 1$, and S_2 the circle centered at $(0, c)$ with radius $r_2 < c - 1$ (see Figure 2). Hence, S_1 can be viewed as a geodesic circle S_H^1 with hyperbolic center $(0, h_1) = (0, \sqrt{1 - r_1^2})$ and hyperbolic radius $\rho_1 = \dfrac{1}{2}\ \ln\Big(\dfrac{1 + r_1}{1 - r_1}\Big)$, and S_2 as a geodesic circle S_H^2 with hyperbolic center $(0, h_2) = (0, \sqrt{c^2 - r_2^2})$ and hyperbolic radius $\rho_2 = \dfrac{1}{2}\ \ln\Big(\dfrac{c + r_2}{c - r_2}\Big)$.

Let β_1 be the central angle of S_H^1 corresponding to the arc above $\{y = 1\}$, and β_2 the central angle of S_H^2 corresponding to the arc below $\{y = c\}$. By (6) we have $\beta_1 = 2\arccos(r_1)$ and $\beta_2 = 2\pi - 2\arccos(r_2/c)$.

For geodesic halfdisks it holds the following result (see Figure 2):

Lemma 3.2.1. *Under the notations above, let $\tilde{S}_1$ be the geodesic through $C_H^1 = (0, h_1)$ and $(r_1, 1)$, and $\tilde{S}_2$ the geodesic through $C_H^2 = (0, h_2)$ and (r_2, c). Let $\theta_1 = \beta_1/2$ and $\theta_2 = \pi - \beta_2/2$, $0 < \theta_1, \theta_2 < \pi/2$. Let S_1^+ be the geodesic halfdisk delimited by S_H^1 and above $\{y = 1\}$, and S_2^- the halfdisk delimited by S_H^2 and below $\{y = c\}$. Then*

$$|\partial S_1^+| = 2\theta_1 \cot\theta_1, \quad |\partial S_2^-| = 2(\pi - \theta_2)\ \cot\theta_2, \tag{7}$$

and

$$|S_1^+| = \frac{2\theta_1}{\sin\theta_1} - \pi + 2\cos\theta_1, \quad |S_2^-| = \frac{2(\pi-\theta_2)}{\sin\theta_2} - \pi - 2\cos\theta_2. \tag{8}$$

Proof: By (2), the arclengths determined by β_1 and β_2 are

$$|\partial S_1^+| = \beta_1 \sinh\rho_1 \quad \text{and} \quad |\partial S_2^-| = \beta_2 \sinh\rho_2.$$

We have

$$\sinh\rho_1 = \sinh\left(\frac{1}{2}\ \ln\left(\frac{1+r_1}{1-r_1}\right)\right) = \frac{r_1}{\sqrt{1-r_1^2}},$$
$$\sinh\rho_2 = \sinh\left(\frac{1}{2}\ \ln\left(\frac{c+r_2}{c-r_2}\right)\right) = \frac{r_2}{\sqrt{c^2-r_2^2}}.$$

Since $\cot\theta_1 = \dfrac{r_1}{\sqrt{1-r_1^2}}$ and $\cot\theta_2 = \dfrac{r_2}{\sqrt{c^2-r_2^2}}$, the first part of the lemma is proved.

Now observe that $|S_1^+|/2 = |\tilde{S}_1| - |\bar{S}_1|$, where $\tilde{S}_1$ is the sector corresponding to θ_1 and $\bar{S}_1$ is the region delimited by $\tilde{S}_1$, axis y and the horocycle $\{y=1\}$. In the same way, $|S_2^-|/2 = |\tilde{S}_2| + |\bar{S}_2|$, where $\tilde{S}_2$ is the sector corresponding to $\beta_2/2 = \pi - \theta_2$ and $\bar{S}_2$ is the region delimited by $\tilde{S}_2$, axis y and the horocycle $\{y=c\}$.

Therefore, by (3)

$$\begin{aligned} |S_1^+| &= 2\theta_1\ (\cosh\rho_1 - 1) - 2(-r_1 + \pi/2 - \arcsin(h_1)), \\ |S_2^-| &= 2(\pi-\theta_2)\ (\cosh\rho_2 - 1) + 2(-r_2/c + \pi/2 - \arcsin(h_2/c)). \end{aligned} \tag{9}$$

But

$$\begin{aligned} \cosh\rho_1 &= \cosh\left(\frac{1}{2}\ \ln\left(\frac{1+r_1}{1-r_1}\right)\right) = \frac{1}{\sqrt{1-r_1^2}}, \\ \cosh\rho_2 &= \cosh\left(\frac{1}{2}\ \ln\left(\frac{c+r_2}{c-r_2}\right)\right) = \frac{c}{\sqrt{c^2-r_2^2}}. \end{aligned} \tag{10}$$

Since $r_1^2 + \left(\sqrt{1-r_1^2}\right)^2 = 1$ and $(r_2/c)^2 + (\sqrt{c^2-r_2^2}/c)^2 = 1$, we have

$$\begin{aligned}
\arccos(r_1) &= \arcsin(\sqrt{1-r_1^2}) = \arcsin(h_1),\\
\arccos(r_2/c) &= \arcsin\left(\frac{\sqrt{c^2-r_2^2}}{c}\right) = \arcsin(h_2/c).
\end{aligned} \tag{11}$$

Furthermore, by (6) it follows that $\cos\theta_1 = r_1$ and $\cos\theta_2 = r_2/c$.

Therefore,

$$\sin\theta_1 = \sqrt{1-r_1^2} \quad \text{and} \quad \sin\theta_2 = \frac{\sqrt{c^2-r_2^2}}{c}. \tag{12}$$

By (9), (10), (11) and (12), the proof of (8) is complete q.e.d.

We observe that a horocycle H can be viewed as a limit geodesic circle with hyperbolic center in $\partial_\infty\mathbb{R}^2_+$. By (6), we have $\cos\theta_1 = r_1$ and the horocycle is obtained when r_1 converges to 1, that is, θ_1 converges to 0. Hence we get the expressions for the perimeter and the area of the horocycle halfdisk as the following consequence of Lemma 3.2.1:

Corollary 3.2.2. *Let H be the horocycle halfdisk above $\{y=1\}$ represented by a Euclidean semicircle with center $(0,1)$ and radius 1. Then*

$$|\partial H| = 2 \quad \textit{and} \quad |H| = 4-\pi. \tag{13}$$

Proof: It is enough to calculate $|\partial S_1^+|$ and $|S_1^+|$ from (7) and (8) for the limit case when $\theta_1 \to 0$ q.e.d.

3.3. Perimeter and area for an equidistant halfdisk

Let $\bar{E}$ be the equidistant curve represented by a Euclidean circle with center $(0,1)$ and radius $r>1$. The Euclidean equation of $\bar{E}$ is given by

$x^2 + (y-1)^2 = r^2$. Then $\bar{E} \cap \partial_\infty \mathbb{R}^2_+ = \{(-\sqrt{r^2-1}, 0), (\sqrt{r^2-1}, 0)\}$. The curve $\bar{E}$ is equidistant from the geodesic η with equation $x^2 + y^2 = r^2 - 1$. If ρ denotes the hyperbolic distance between $\bar{E}$ and η, then ρ is the hyperbolic distance between $(0, 1+r)$ and $(0, \sqrt{r^2-1})$, so that $\rho = \ln\left(\frac{r+1}{r-1}\right)^{\frac{1}{2}}$, whence

$$r = \coth \rho. \tag{14}$$

If α is the non-oriented angle between $\bar{E}$ and η, $0 < \alpha < \pi/2$, then (for instance, see Proposition 3 in Chapter 5 of [4])

$$\tanh \rho = \sin \alpha. \tag{15}$$

Lemma 3.3.1. *Under the notations above, let E be the equidistant halfdisk above $\{y = 1\}$. Then*

$$\begin{aligned} |\partial E| &= \frac{2}{\cos\alpha} \ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right), \\ |E| &= \frac{2}{\sin\alpha} - \pi + \frac{2}{\cot\alpha} \ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right). \end{aligned} \tag{16}$$

Proof: In order to calculate $|\partial E|$, we parametrize E by

$$\beta(t) = (r\cos t, 1 + r\sin t), \ 0 \le t \le \pi.$$

Then

$$|\partial E| = 2\int_0^{\pi/2} \frac{r}{1 + r\sin t}\, dt = \frac{2r}{\sqrt{r^2-1}} \ \ln\left(\frac{r + \tan(t/2) - \sqrt{r^2-1}}{r + \tan(t/2) + \sqrt{r^2-1}}\right)\Bigg|_0^{\pi/2}$$

$$= \frac{2r}{\sqrt{r^2-1}} \ \ln(r + \sqrt{r^2-1}).$$

By (14) and (15) we have $r = 1/\sin\alpha$, whence $\sqrt{r^2-1} = \cot\alpha$, because $0 < \alpha < \pi/2$. Therefore, $|\partial E| = \frac{2}{\cos\alpha} \ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right)$, and the first part

of (16) is proved. Now,

$$|E| = 2\int_0^r \int_1^{1+\sqrt{r^2-x^2}} \frac{1}{y^2}\, dy\, dx = 2r - \pi + \frac{1}{\sqrt{r^2-1}} \ln\left|\frac{r\sqrt{r^2-1}+(r^2-1)}{r\sqrt{r^2-1}-(r^2-1)}\right|.$$

By (14) and (15), it follows that $|E|$, as function of the equidistance angle α, is given by $|E| = \dfrac{2}{\sin\alpha} - \pi + \dfrac{2}{\cot\alpha}\ \ln\Big(\dfrac{1}{\sin\alpha} + \cot\alpha\Big)$, which proves the second part of (16) q.e.d.

4. Comparison of perimeters of regions with prescribed area

In this section we analyze the expressions of the perimeter and area of the regions delimited by curves of constant geodesic curvature. Their isoperimetric profiles in $\mathcal{F}_c$ will be obtained in the next section as functions of its hyperbolic width d. Since we have been taking the horocycles $\{y = 1\}$ and $\{y = c\}$, the constant c must satisfy the following condition: if H is a horocycle halfdisk above $\{y = c\}$ and T is a section in $\mathcal{F}_c$, then $|\partial H| = |\partial T|$. By (13), this means $2 = 2\ln c$, whence $c = e$ and $d = 1$. This is why we compare d with 1 in Theorem 1.1.

Let S_1 be a Euclidean circle with radius r_1 and center $(0, 1)$ above $\{y = 1\}$. When $0 < r_1 < 1$, S_1 delimits a geodesic halfdisk. Consider the limit cases $\theta_1 \to 0$ and $\theta_1 \to \pi/2$, which correspond to a horocycle halfdisk ($r_1 \to 1$) and a point ($r_1 \to 0$), respectively. From (6) and (7) we have

$$\lim_{\theta_1\to 0} |\partial S_1^+| = 2, \quad \lim_{\theta_1\to\pi/2} |\partial S_1^+| = 0, \tag{17}$$

and from (1),

$$\lim_{\theta_1\to 0} |S_1^+| = 4-\pi, \quad \lim_{\theta_1\to\pi/2} |S_1^+| = 0. \tag{18}$$

Therefore, $|\partial S_1^+|$ increases from 0 to 2 when r_1 varies from 0 to 1, while $|S_1^+|$ increases from 0 to $4-\pi$.

If $r_1 > 1$ we have an equidistant halfdisk. By (14) and (15) the limit cases are obtained for $\alpha \to \pi/2$ (that is, $r_1 \to 1$) and $\alpha \to 0$ (that is, $r_1 \to \infty$). By (16),

$$\lim_{\alpha\to\pi/2} |\partial E| = 2, \quad \lim_{\alpha\to 0} |\partial E| = \infty, \tag{19}$$

and

$$\lim_{\alpha\to\pi/2} |E| = 4-\pi, \quad \lim_{\alpha\to 0} |E| = \infty. \tag{20}$$

Therefore, both $|\partial E|$ and $|E|$ increase infinitely while r_1 increases.

Since $c > 1$, let S_2 be a Euclidean semicircle with radius r_2 and center $(0, c)$ below $\{y = c\}$. When $0 < r_2 < c$, S_2 delimits a geodesic halfdisk. By (6), the limit cases $\theta_2 \to 0$ and $\theta_2 \to \pi/2$ correspond to $r_2 \to c$ and $r_2 \to 0$, respectively. By (7), we have for these limit cases

$$\lim_{\theta_2\to 0} |\partial S_2^-| = \infty, \quad \lim_{\theta_2\to\pi/2} |\partial S_2^-| = 0, \tag{21}$$

and by (8),

$$\lim_{\theta_2\to 0} |S_2^-| = \infty, \quad \lim_{\theta_2\to\pi/2} |S_2^-| = 0. \tag{22}$$

Then we see that $|\partial S_2^-|$ and $|S_2^-|$ increase infinitely while $r_2 \to c$. If $r_2 \geq c$, S_2 delimits a horocycle halfdisk or an equidistant halfdisk. By (21) and (22), if $r_2 \geq c$, then both $|\partial S_2^-|$ and $|S_2^-|$ diverge to infinity.

From the analysis we have just done for the perimeter and area of the possible isoperimetric solutions, there are only the following cases to consider:

1. to compare a geodesic halfdisk above $\{y = 1\}$ with a geodesic disk entirely contained in $\mathcal{F}_c$;

2. to compare a geodesic halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$;
3. to compare a horocycle halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$;
4. to compare an equidistant halfdisk above $\{y = 1\}$ with a geodesic halfdisk below $\{y = c\}$.

In order to prove the second part of Theorem 1.1, one must determine the least-perimeter regions with prescribed area. For this purpose, we will use a strategy: we determine the regions with prescribed perimeter and biggest area. In fact, it is enough to show that if a region has the maximum area among all regions with a prescribed perimeter, then it has the minimum perimeter among all regions with the same prescribed area (see Lemma 4.1 below). Since we have just listed all possible isoperimetric solutions besides the section, Lemma 4.1 will then refer to the above case 2. The other cases are proved analogously. Without loss of generality we suppose that the geodesic halfdisk above $\{y = 1\}$ has maximum area when compared to any geodesic halfdisk below $\{y = c\}$ with the same perimeter.

Lemma 4.1. *Let Ω_0 be the geodesic halfdisk above $\{y = 1\}$ with $|\Omega_0| \geq |\Omega|$, whenever $|\partial\Omega| = |\partial\Omega_0|$, for any geodesic halfdisk Ω below $\{y = c\}$, $c > 1$. If Ω_1 is a geodesic halfdisk below $\{y = c\}$ with $|\Omega_0| = |\Omega_1|$, then $|\partial\Omega_0| \leq |\partial\Omega_1|$.*

Proof: Suppose by contradiction that $|\partial\Omega_0| > |\partial\Omega_1|$. By (21) we can increase the radius of the Euclidean circle that represents Ω_1 till we get a geodesic halfdisk Ω' such that $|\partial\Omega'| = |\partial\Omega_0|$. This procedure could fail if Ω' surpassed $\{y = 1\}$, but then the section will prevail as the isoperimetric

solution. This fact will be proved later on in Section 5. By (22), the area increases with the radius. Therefore, $|\Omega'| > |\Omega_1| = |\Omega_0|$ and $|\partial\Omega'| = |\partial\Omega_0|$. This is a contradiction with the fact that Ω_0 maximizes the area when compared to regions of the same perimeter, by hypothesis q.e.d.

Till the end of this section we are going to compare the area of the possible isoperimetric solutions for a prescribed perimeter.

For case 1 described above, we compare the area of a geodesic halfdisk above $\{y = 1\}$ with a geodesic disk entirely contained in $\mathcal{F}_c$, when they have the same perimeter. Let S be the Euclidean circle with radius r_2, $0 < r_2 < y_2 - 1$, and center $(0, y_2)$, $1 < y_2 < c$, which delimits the geodesic halfdisk (see Figure 4).

Consider $\theta_2 \in \,]0, \pi/2[$ such that $\cos\theta_2 = r_2/c$. By (2), (3) and (4), if $\mathcal{S}$ is the geodesic disk corresponding to a central angle of 2π then $|\partial\mathcal{S}| = 2\pi\cot\theta_2$ and $|\mathcal{S}| = \dfrac{2\pi}{\sin\theta_2} - 2\pi$.

By (7), (8) and the information from the previous paragraph, we show that $|S_1^+| > |\mathcal{S}|$ when $|\partial S_1^+| = |\partial\mathcal{S}|$ in the next Lemma.

Lemma 4.2. *Let $\theta_1, \theta_2 \in \,]0, \pi/2[$ such that*

$$\theta_1 \cot\theta_1 = \pi\cot\theta_2. \tag{23}$$

Then

$$\frac{2\theta_1}{\sin\theta_1} + 2\cos\theta_1 - \pi > \frac{2\pi}{\sin\theta_2} - 2\pi. \tag{24}$$

Proof: For $\theta_2 \in \,]0, \pi/2[$, by calculating the squares of (23) and using that $\cos^2\theta_2 = 1 - \sin^2\theta_2$, one has $1/\sin\theta_2 = \sqrt{\theta_1^2\cot^2\theta_1 + \pi^2}/\pi$. Thus

$$\frac{2\pi}{\sin\theta_2} - 2\pi = 2\sqrt{\left(\frac{\theta_1}{\sin\theta_1}\right)^2 - \theta_1^2 + \pi^2} - 2\pi. \tag{25}$$

Now we replace the right-hand side of (24) by (25), and define

$$A(\theta_1) := \frac{2\theta_1}{\sin\theta_1} + 2\cos\theta_1 - \pi - 2\sqrt{\left(\frac{\theta_1}{\sin\theta_1}\right)^2 - \left(\theta_1\right)^2 + \pi^2} + 2\pi,$$

so that (24) will hold if and only if $A(\theta_1) > 0$.

We observe that

$$\lim_{\theta_1 \to 0} A(\theta_1) = 4 + \pi - 2\sqrt{\pi^2 + 1} > 0 \quad \text{and} \quad \lim_{\theta_1 \to \pi/2} A(\theta_1) = 0. \qquad (26)$$

Moreover,

$$\frac{dA(\theta_1)}{d\theta_1} = \frac{2\cos\theta_1\Big(\sin\theta_1\cos\theta_1 - \theta_1\Big)}{\sin^2\theta_1}\left\{1 - \frac{\theta_1}{\sqrt{\theta_1^2 + \left(\pi^2 - \theta_1^2\right)\sin^2\theta_1}}\right\} < 0,$$

because $\theta_1 \in \,]0, \pi/2[$ implies $\cos\theta_1 > 0$, $\sin\theta_1\cos\theta_1 - \theta_1 < 0$ and

$$0 < \frac{\theta_1}{\sqrt{\theta_1^2 + \left(\pi^2 - \theta_1^2\right)\sin^2\theta_1}} < 1.$$

Therefore, $A(\theta_1)$ decreases in $]0, \pi/2[$. By (26), we conclude that $A(\theta_1) > 0$ in $]0, \pi/2[$, whence (24) is proved q.e.d.

We conclude from Lemma 4.2 that the geodesic halfdisk above $\{y = 1\}$ is the isoperimetric solution, instead of the geodesic disk, which concludes case 1.

Now we study case 2. By (7) and (8), we will show in the next Lemma and Corollary that $|S_1^+| > |S_2^-|$ when $|\partial S_1^+| = |\partial S_2^-|$. In Figure 4, the dashed circle was obtained from the lower by a Euclidean homothety so that the corresponding geodesic halfdisks have the same perimeter. By (5), in order to have $|\partial S_1^+| = |\partial S_2^-|$, it is necessary to decrease the radius of S_2^-.

Lemma 4.3. *Let* $\theta_1, \theta_2 \in]0, \pi/2]$ *such that*

$$\theta_1 \cot \theta_1 = (\pi - \theta_2) \cot \theta_2. \tag{27}$$

Then

$$\frac{\theta_1}{\sin \theta_1} + \cos \theta_1 \geq \frac{\pi - \theta_2}{\sin \theta_2} - \cos \theta_2. \tag{28}$$

Proof: For $\theta_1, \theta_2 \in]0, \pi/2]$, we define $f(\theta_1) = \dfrac{\theta_1}{\sin \theta_1} + \cos \theta_1$ and $F(\theta_1, \theta_2) = f(\theta_1) + f(\theta_2) - \dfrac{\pi}{\sin \theta_2}$. We want to show that $F(\theta_1, \theta_2) \geq 0$. By (27) we can define θ_1 implicitly as a function of θ_2. Namely, we get a function g such that $\theta_1 = g(\theta_2)$. Let $h_1(\theta_2) = F(g(\theta_2), \theta_2)$ and $h_2(\theta_2) = h_1(\theta_2) \sin \theta_2 + \pi = \sin \theta_2 \ f(g(\theta_2)) + \sin \theta_2 \ f(\theta_2)$.

The function $h_2(\theta_2)$ is $\mathcal{C}^\infty$ and

$$h_2'(\theta_2) = \cos \theta_2 \ f(\theta_1) + \sin \theta_2 \ f'(\theta_1) \ g'(\theta_2) + \cos \theta_2 \ f(\theta_2) + \sin \theta_2 \ f'(\theta_2). \tag{29}$$

Hence

$$f'(\theta_1) = \frac{\sin \theta_1 - \theta_1 \cos \theta_1}{\sin^2 \theta_1} - \sin \theta_1 = \frac{-\cos \theta_1 (2\theta_1 - \sin 2\theta_1)}{2 \sin^2 \theta_1}. \tag{30}$$

From the Implicit Function Theorem we have

$$g'(\theta_2) = \frac{-\cot \theta_2 - (\pi - \theta_2) \csc^2 \theta_2}{\cot \theta_1 - \theta_1 \csc^2 \theta_1}.$$

Now observe that $\cot \theta_1 - \theta_1 \csc^2 \theta_1 = \dfrac{\sin 2\theta_1 - 2\theta_1}{2 \sin^2 \theta_1}$. Therefore,

$$g'(\theta_2) = \frac{2(\pi - \theta_2) + \sin 2\theta_2}{2 \sin^2 \theta_2} \ \frac{2 \sin^2 \theta_1}{2\theta_1 - \sin 2\theta_1}. \tag{31}$$

By substituting (30) and (31) in (29), we obtain

$$h_2'(\theta_2) = 2 \cos^2 \theta_2 + \frac{\theta_1 \cos \theta_2}{\sin \theta_1} - \frac{(\pi - \theta_2) \cos \theta_1}{\sin \theta_2}. \tag{32}$$

Since $h_2'(\theta_2) = h_1'(\theta_2)\sin\theta_2 + h_1(\theta_2)\cos\theta_2$, it follows from (32) that

$$h_1'(\theta_2)\sin\theta_2 = 2\cos^2\theta_2 + \frac{\theta_1\cos\theta_2}{\sin\theta_1} - \frac{(\pi-\theta_2)\cos\theta_1}{\sin\theta_2} - F(\theta_1,\theta_2)\cos\theta_2$$

$$= \cos^2\theta_2 - \cos\theta_2\ \cos\theta_1 - \frac{(\pi-\theta_2)}{\sin\theta_2}(\cos\theta_1 - \cos\theta_2)$$

$$= (\cos\theta_2 - \cos\theta_1)(\cos\theta_2 + \frac{\pi-\theta_2}{\sin\theta_2}). \tag{33}$$

For $\theta_1, \theta_2 \in\ ]0, \pi/2]$, we have $\theta_1 \leq \pi - \theta_2$, which by (27) implies

$$\cot\theta_1 = \left(\frac{\pi-\theta_2}{\theta_1}\right)\cot\theta_2 \geq \cot\theta_2 \Rightarrow \theta_1 \leq \theta_2 \Rightarrow \cos\theta_1 \geq \cos\theta_2. \tag{34}$$

By (33) and (34) we have $h_1'(\theta_2) \leq 0$. Thus $F(\theta_1,\theta_2) = F(g(\theta_2),\theta_2) = h_1(\theta_2)$ is a decreasing function for $\theta_2 \in\ ]0,\pi/2]$. By (27), for $\theta_2 = \pi/2$ and $\theta_1 = g(\pi/2)$, one has $\cos(g(\pi/2)) = 0$ and therefore $g(\pi/2) = \pi/2$. Since

$$h_1(\pi/2) = F(g(\pi/2), \pi/2) = F(\pi/2, \pi/2) = 0,$$

then $F(\theta_1,\theta_2) \geq F(g(\pi/2),\pi/2) = 0$. Consequently, $f(\theta_1) + f(\theta_2) \geq \dfrac{\pi}{\sin\theta_2}$, whence (28) is proved.

The equality occurs if and only if $\theta_1 = \theta_2 = \pi/2$.

q.e.d.

Corollary 4.4. *Let $\theta_1, \theta_2 \in\ ]0, \pi/2[$ such that*

$$\theta_1\cot\theta_1 = (\pi - \theta_2)\cot\theta_2. \tag{35}$$

Then

$$\frac{\theta_1}{\sin\theta_1} + \cos\theta_1 > \frac{\pi-\theta_2}{\sin\theta_2} - \cos\theta_2. \tag{36}$$

We finally conclude from Corollary 4.4 that the geodesic halfdisk above $\{y = 1\}$ is the isoperimetric solution for case 2.

By (13), (7) and (8), we show in the next Lemma that $|H| > |S_2^-|$ when $|\partial H| = |\partial S_2^-|$. Case 3 is illustrated in Figure 5.

Lemma 4.5. *Let* $\theta_2 \in\ ]0, \pi/2[$ *such that* $1 = (\pi - \theta_2)\cot\theta_2$. *Then* $2 > \dfrac{\pi - \theta_2}{\sin\theta_2} - \cos\theta_2$.

Proof: Since the horocycle is obtained from the geodesic halfdisk above $\{y = 1\}$ when $\theta_1 \to 0$, then it is enough to make $\theta_1 \to 0$ in (35) and (36). The result follows from the continuity of the involved functions q.e.d.

We conclude from Lemma 4.5 that the horocycle halfdisk above $\{y = 1\}$ is the isoperimetric solution, instead of the geodesic halfdisk below $\{y = c\}$.

Now we analyze case 4. By (16), (7) and (8), we show in the next Lemma that $|E| > |S_2^-|$ when $|\partial E| = |\partial S_2^-|$. In Figure 5, the dashed circle was obtained from the lower by a Euclidean homothety so that they have the same perimeter. In order to have $|\partial E| = |\partial S_2^-|$, it is necessary to decrease the radius of S_2^-.

Lemma 4.6. *Let* $\alpha, \theta_2 \in\ ]0, \pi/2[$ *such that*

$$\frac{1}{\cos\alpha}\ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right) = (\pi - \theta_2)\cot\theta_2. \tag{37}$$

Then

$$\frac{1}{\sin\alpha} + \frac{1}{\cot\alpha}\ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right) \geq \frac{\pi - \theta_2}{\sin\theta_2} - \cos\theta_2. \tag{38}$$

Proof: For $\alpha, \theta_2 \in\ ]0, \pi/2[$ we define

$$F(\alpha, \theta_2) = \frac{1}{\sin\alpha} + \frac{1}{\cot\alpha}\ \ln\left(\frac{1}{\sin\alpha} + \cot\alpha\right) + \frac{\theta_2}{\sin\theta_2} + \cos\theta_2 - \frac{\pi}{\sin\theta_2}.$$

By (37), we can implicitly define θ_2 as a function of α. Namely, one gets a function g such that $\theta_2 = g(\alpha)$. Consider the functions $h_1(\alpha) = F(\alpha, g(\alpha))$,

$h_2(\alpha) = h_1(\alpha)\sin\alpha$. Then

$$h_2(\alpha) = \Big(\frac{1}{\sin\alpha} + \frac{1}{\cot\alpha}\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big)\Big)\sin\alpha - \Big(\frac{\pi - g(\alpha)}{\sin g(\alpha)} - \cos g(\alpha)\Big)\ \sin\alpha.$$

The function $h_2(\alpha)$ is $\mathcal{C}^\infty$ and

$$h_2'(\alpha) = \frac{\sin\alpha}{\cos^2\alpha}\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) - \frac{\sin\alpha}{\cos\alpha} + \sin\alpha\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) +$$

$$+ g'(\alpha)\ \frac{\cos\theta_2(\sin(2\theta_2) + 2(\pi - \theta_2))}{2\sin^2\theta_2}\sin\alpha - \frac{(\pi - \theta_2)}{\sin\theta_2}\cos\alpha + \cos\theta_2\cos\alpha. \tag{39}$$

From the Implicit Function Theorem we have

$$g'(\alpha) = -\frac{\dfrac{\sin\alpha}{\cos^2\alpha}\ \ln\Big(\dfrac{1}{\sin\alpha} + \cot\alpha\Big) - \dfrac{1}{\sin\alpha\cos\alpha}}{\cot\theta_2 + (\pi - \theta_2)\csc^2\theta_2}.$$

Since $\cot\theta_2 + (\pi - \theta_2)\csc^2\theta_2 = \dfrac{\sin 2\theta_2 + 2(\pi - \theta_2)}{2\sin^2\theta_2}$, then

$$g'(\alpha) = \Big(\frac{2\sin^2\theta_2}{\sin 2\theta_2 + 2(\pi - \theta_2)}\Big)\ \Big(\frac{1}{\sin\alpha\cos\alpha} - \frac{\sin\alpha}{\cos^2\alpha}\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big)\Big). \tag{40}$$

By substituting (40) in (39), we obtain

$$h_2'(\alpha) = \frac{\sin\alpha}{\cos^2\alpha}\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) - \frac{\sin\alpha}{\cos\alpha} + \sin\alpha\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) +$$

$$+ \frac{\cos\theta_2}{\cos\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha}\ \cos\theta_2\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) - \frac{(\pi - \theta_2)}{\sin\theta_2}\cos\alpha + \cos\theta_2\cos\alpha. \tag{41}$$

Since $h_2'(\alpha) = h_1'(\alpha)\sin\alpha + h_1(\alpha)\cos\alpha$, it results from (41) that

$$h_1'(\alpha)\sin\alpha = \frac{1 - \sin\alpha\ \cos\theta_2}{\cos\alpha}\ \Big(\tan\alpha\ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) - \frac{1}{\sin\alpha}\Big).$$

For $\alpha \in\ ]0, \pi/2[$, if $l(\alpha) = \tan\alpha\ \ln(1/\sin\alpha + \cot\alpha)$, then $l'(\alpha) = \sec^2\alpha\ k(\alpha)$, where $k(\alpha) = \ln(1/\sin\alpha + \cot\alpha) - \cos\alpha$. Since $k'(\alpha) = -\cos^2\alpha/\sin\alpha < 0$,

then $k(\alpha)$ is decreasing in $]0,\pi/2[$ and $\lim\limits_{\alpha\to\pi/2} k(\alpha) = 0$. So $k(\alpha) > 0$ in $]0,\pi/2[$. Consequently, $l(\alpha)$ is increasing in $]0,\pi/2[$. Since $\lim\limits_{\alpha\to\pi/2} l(\alpha) = 1$, then $l(\alpha) < 1$ and therefore

$$\tan\alpha \ \ln\Big(\frac{1}{\sin\alpha} + \cot\alpha\Big) - \frac{1}{\sin\alpha} < 0.$$

Moreover, for $\alpha, \theta_2 \in \ ]0,\pi/2[$ we have $0 < \sin\alpha\cos\theta_2 < 1$. Thus, for $\alpha \in \ ]0,\pi/2[$ we conclude that $h_1'(\alpha)\sin\alpha < 0$ and therefore $h_1'(\alpha) < 0$. Namely, h_1 is decreasing in $]0,\pi/2[$. In particular,

$$h_1(\alpha) \geq \lim_{\alpha\to\pi/2} h_1(\alpha) = \lim_{\alpha\to\pi/2} F(\alpha, g(\alpha)) = 2 - \Big(\frac{\pi-\beta}{\sin\beta} - \cos\beta\Big),$$

where $\beta = \lim\limits_{\alpha\to\pi/2} g(\alpha)$. From (13) it follows that

$$\frac{|H| + \pi}{2} = 2,$$

where H is a horocycle halfdisk above $\{y = 1\}$. By (8),

$$\frac{|G| + \pi}{2} = \frac{\pi-\beta}{\sin\beta} - \cos\beta,$$

where G is a geodesic halfdisk with the same perimeter as H (just take $\alpha \to \pi/2$ in (37) and $\theta_2 = \beta$). But from Lemma 4.5 we conclude that

$$h_1(\alpha) = 2 - \Big(\frac{\pi-\beta}{\sin\beta} - \cos\beta\Big) > 0,$$

whence (38) is proved q.e.d.

From Lemma 4.2, Corollary 4.4, Lemma 4.5 and Lemma 4.6, we conclude that the family of geodesic, horocycle and equidistant halfdisks above $\{y = 1\}$ are the solutions to the isoperimetric problem, instead of the geodesic halfdisks below $\{y = c\}$, $\ c > 1$.

5. Isoperimetric Profile in $\mathbb{R}^2_+$

In this section we study the isoperimetric profile for $\mathcal{F}_c$ (see Figure 6). We adapt a well-known result from the Isoperimetric Problem Theory which guarantees that the boundaries of the connected components of an isoperimetric solution are curves with the same constant geodesic curvature (for instance, see Lemma 2.1 of [1]). Before showing that a minimizing region is made up with a single connected component, we prove that a connected component of an isoperimetric region *must* be either a section or a halfdisk above the horocycle $\{y = 1\}$. Here we need (17)-(22). The perimeter of the section in $\mathcal{F}_c$ is equal to $2 \ln c$. Now there are only three possibilities that we classify according to the hyperbolic distance $d = \ln c$.

First Possibility: $d < 1$

1. Consider a horocycle $\{y = c\}$ with $1 < c < e$. Let $A_0(c)$ be the area of the geodesic halfdisk S_0 above $\{y = 1\}$, centered at $(0, 1)$ with Euclidean radius $r_0(c)$ and $|\partial S_0| = |\partial T_0|$, where T_0 is a section with $|T_0| = A_0(c)$ (see Figure 7). Since $c < e$, then $|\partial T_0| < 2$ (which is the perimeter of the horocycle halfdisk above $\{y = 1\}$).

   Consequently,

   - if $A = A_0(c)$ then $|\partial S_0| = |\partial T_0|$ and $|S_0| = |T_0| = A$. Therefore, the minimizing region Ω is a geodesic halfdisk or a section;
   - if $A < A_0(c)$, let S_1 be a geodesic halfdisk with area A, centered at $(0, 1)$ and with Euclidean radius r_1. Since both $|S_1|$ and $|\partial S_1|$

decrease with r_1, we have $r_1 < r_0(c)$ and $|\partial S_1| < |\partial S_0|$. Let T_1 be a section with $|T_1| = A$. Then $|S_1| = |T_1| = A$, but $|\partial S_1| < |\partial T_1| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a geodesic halfdisk. In this case, we observe that $|\Omega| = A < |S_0|$, so that Ω can neither be a horocycle nor an equidistant halfdisk;

- if $A > A_0(c)$, let S_2 be a geodesic halfdisk with $|S_2| = A$, centered at $(0, 1)$ and with Euclidean radius r_2. Since both $|S_2|$ and $|\partial S_2|$ increase with r_2, then $r_2 > r_0(c)$ and $|\partial S_2| > |\partial S_0|$. Let T_2 be a section with $|T_2| = A$. Then $|S_2| = |T_2| = A$, but $|\partial S_2| > |\partial T_2| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a section.

Second Possibility: $d = 1$

2. Suppose $d = 1$. Consider the horocycle $\{y = c\}$ with $c = e$. Then $A_0(c) = 4 - \pi$ is the area of the horocycle halfdisk S_0 above $\{y = 1\}$, centered at $(0, 1)$ with Euclidean radius $r_0(c) = 1$ and $|\partial S_0| = |\partial T_0|$, where T_0 is a section with $|T_0| = A_0(c)$ (see Figure 8). In this case, $|\partial T_0| = 2$.

Consequently,

- if $A = A_0(c)$, then $|S_0| = |T_0| = A$. Therefore, the minimizing Ω is a horocycle halfdisk or a section;

- if $A < A_0(c)$, let S_1 be a geodesic halfdisk with $|S_1| = A$, centered at $(0, 1)$ and with Euclidean radius r_1. Since both $|S_1|$ and $|\partial S_1|$ increase with r_1 till it becomes a horocycle disk, then $r_1 < 1$ and $|\partial S_1| < |\partial S_0|$. Let T_1 be a section with $|T_1| = A$. Then

$|S_1| = |T_1| = A$, but $|\partial S_1| < |\partial T_1| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a geodesic halfdisk;

- if $A > A_0(c)$, let S_2 be an equidistant halfdisk $|S_2| = A$, centered at $(0,1)$ and with Euclidean radius r_2. Since both $|S_2|$ and $|\partial S_2|$ increase infinitely with r_2, then $r_2 > 1$ and $|\partial S_2| > |\partial S_0|$. Let T_2 be a section with $|T_2| = A$. Then $|S_2| = |T_2| = A$, but $|\partial S_2| > |\partial T_2| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a section.

Third Possibility: $d > 1$

3. Suppose $d > 1$. Consider a horocycle $\{y = c\}$ with $c > e$. Let $A_0(c) = 4 - \pi$ be the area of the horocycle halfdisk S_0 above $\{y = 1\}$, centered at $(0,1)$ with Euclidean radius $r_0(c) = 1$ and $|\partial S_0| = 2$. Let T_0 be a section with $|T_0| = A_0(c)$ and $A_1(c)$ be the area of an equidistant halfdisk S_1 above $\{y = 1\}$, centered at $(0,1)$ with Euclidean radius $r_1(c)$ and $|\partial S_1| = |\partial T_1|$, where T_1 is a section with $|T_1| = A_1(c)$ (see Figure 9). In this case, we observe that $|\partial T_1| > 2$.

   Consequently,

   - if $A = A_0(c) = 4 - \pi$ then $|S_0| = |T_0| = A$, but $|\partial T_0| > 2 = |\partial S_0|$. Therefore, the minimizing Ω is a horocycle halfdisk;
   - if $A = A_1(c)$ then $|S_1| = |T_1| = A$ and $|\partial S_1| = |\partial T_1|$. Therefore, the minimizing Ω is an equidistant halfdisk or a section;
   - if $A < A_0(c)$, let S_2 be a geodesic halfdisk with $|S_2| = A$, centered at $(0,1)$ and with Euclidean radius r_2. Then $r_2 < r_0(c)$ and $|\partial S_2| < |\partial S_0|$. Let T_2 be a section with $|T_2| = A$. Then

$|S_2| = |T_2| = A$, but $|\partial S_2| < |\partial T_2| = |\partial T_0| = |\partial S_0|$. Therefore, the minimizing Ω is a geodesic halfdisk;

- if $A_0(c) < A < A_1(c)$, let S_3 be an equidistant halfdisk with $|S_3| = A$, centered at $(0,1)$ and with Euclidean radius r_3. Then $r_0(c) < r_3 < r_1(c)$ and $|\partial S_3| < |\partial S_1|$. Let T_3 be a section with $|T_3| = A$. Then $|S_3| = |T_3| = A$, but $|\partial S_3| < |\partial T_3| = |\partial T_1| = |\partial S_1|$. Therefore, the minimizing Ω is an equidistant halfdisk;
- if $A > A_1(c)$, let S_4 be an equidistant halfdisk with $|S_4| = A$, centered at $(0,1)$ and with Euclidean radius r_4. Then $r_4 > r_1(c)$ and $|\partial S_4| > |\partial S_1|$. Let T_4 be a section with $|T_4| = A$. Then $|S_4| = |T_4| = A$, but $|\partial S_4| > |\partial T_4| = |\partial T_1| = |\partial S_1|$. Therefore, the minimizing Ω is a section.

REMARK 5.1: A minimizing region consists of only one connected component, and in fact it is enough to show that it can not have two. If this were the case, their geodesic curvatures would agree. Consider $A > 0$ and Ω' a region with area A and two disjoint sections. Their "gluing" would result in another section with area A but with smaller perimeter, because two vertical geodesics would not count anymore. Then Ω' is not minimizing.

The other case to consider is two connected components consisting of two geodesic halfdisks above $\{y = 1\}$. In this case, we use the fact that a non-regular region is not minimizing: let $A > 0$ and Ω' be a region with area A and two geodesic halfdisks above $\{y = 1\}$ with the same Euclidean radius, hence the same geodesic curvature. By sliding one of them over $\{y = 1\}$ till it touches the other, since horizontal translations are isometries of the hyperbolic plane, we get a non-regular region Ω'' with area A. Then Ω'' does

not have the least-perimeter among all regions with prescribed area A. Since $|\Omega'| = |\Omega''|$, Ω' is not minimizing.

Therefore, a minimizing region must consist of a single connected component.

Now we prove Theorem 1.1.

Proof: The first part of Theorem 1.1 was already discussed in the Preliminaries. The existence of such an isoperimetric region follows from adaptions of some results from [5] and [6]: the group G of isometries of $\mathbb{R}^2_+$ that leave $\mathcal{F}_c$ invariant consists of horizontal Euclidean translations and Euclidean reflections with respect to a vertical geodesic, so that $\mathcal{F}_c/G$ is homeomorphic to the interval $[0, 1]$, hence compact.

The second part of Theorem 1.1 follows from the analysis of the isoperimetric profile done in the three possibilities above, together with REMARK 5.1.

References

[1] C. Adams and F. Morgan, Isoperimetric curves on hyperbolic surfaces. Proc. Am. Math. Soc. 127 (1986), 1347–1356.

[2] M. Athanassenas, A variational problem for constant mean curvature surfaces with free boundary. J. Reine Angew. Math. 377 (1986), 97–107.

[3] R.M.B. Chaves, M.F. da Silva and R.H.L. Pedrosa, A free boundary isoperimetric problem in the hyperbolic space between parallel horospheres. Pre-print at `http://arxiv.org/abs/0811.1046v1`

[4] R.S. Earp and E. Toubiana, Cours de Geometrie Hyperbolique et de Surfaces de Riemann. PUC-RJ, Monografias 1, 1996.

[5] F. Morgan, Clusters minimizing area plus length of singular curves. Math. Ann. 299 (1994), 697–714.

[6] F. Morgan, Geometric Measure Theory (A Beginner's Guide). Academic Press, 4th edition, San Diego, 2009.

[7] M. Simonson, The isoperimetric problem on Euclidean, spherical, and hyperbolic surfaces. Senior honor thesis, Williams College, 2008.

[8] T. Vogel, Stability of a drop trapped between two parallel planes. SIAM J. Appl. Math. 47 (1987), 1357–1394.

Márcio Fabiano da Silva
Universidade Federal do ABC
r. Catequese 242, 3rd floor
09090-400 Santo André - SP, Brazil
marcio.silva@ufabc.edu.br

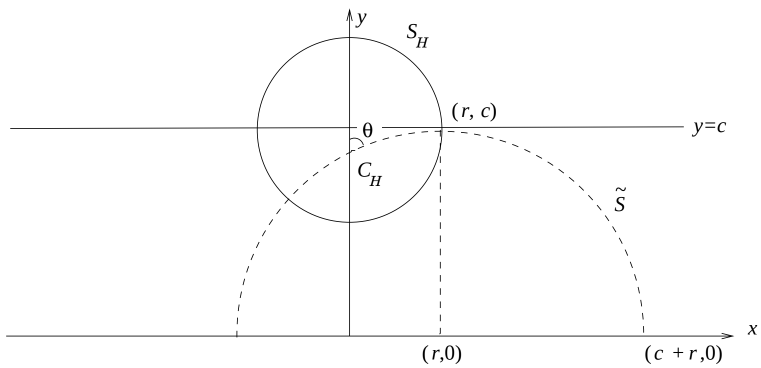

Figure 1: Arc of geodesic circle corresponding to a central angle θ.

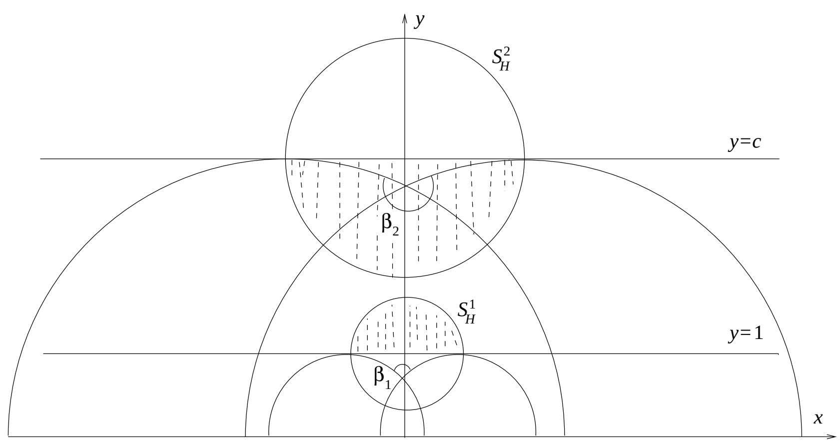

Figure 2: Perimeter and area for geodesic halfdisks.

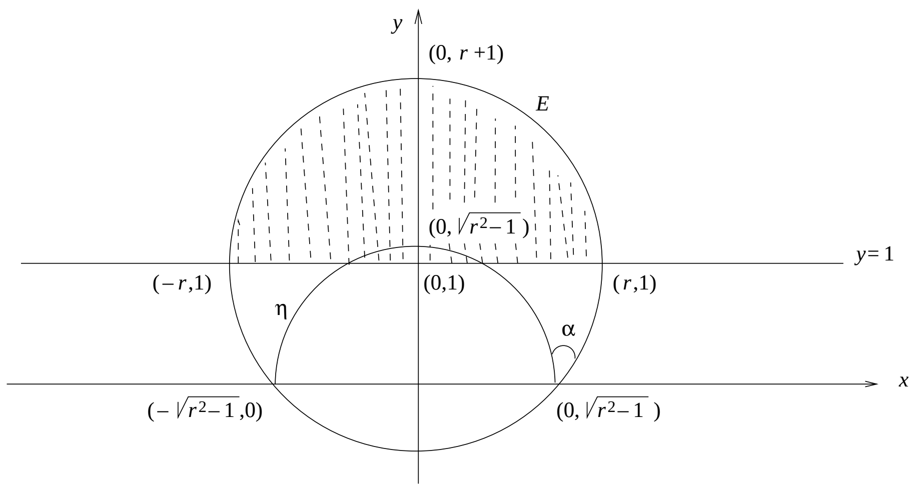

Figure 3: Perimeter and area for an equidistant disk.

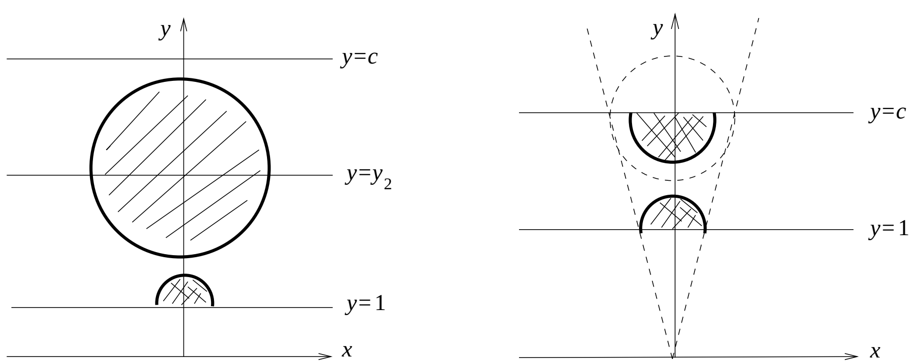

Figure 4: Cases 1 (left) and 2 (right).

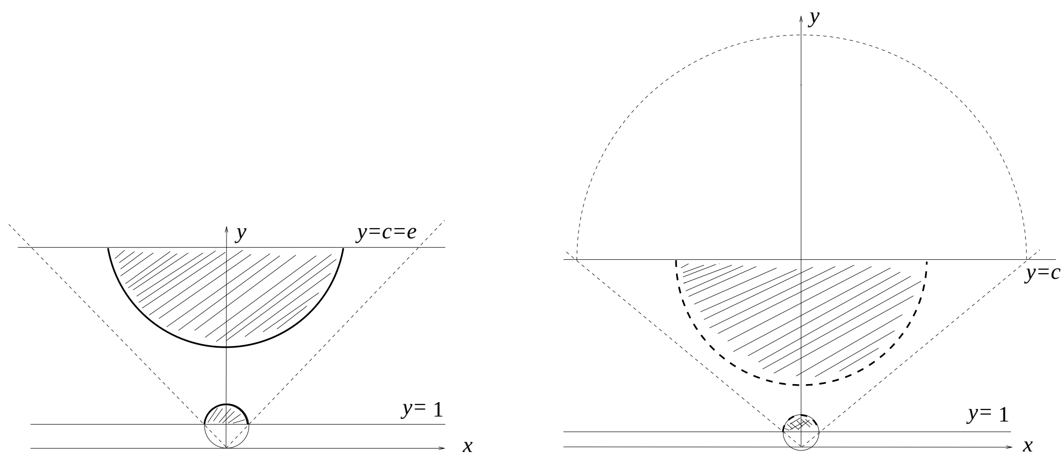

Figure 5: Cases 3 (left) and 4 (right).

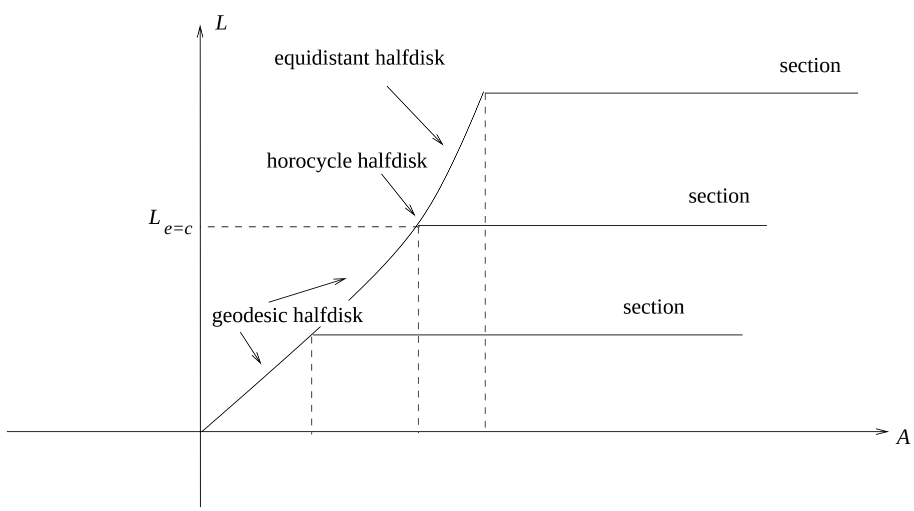

Figure 6: Isoperimetric profile for the region between the parallel horocycles.

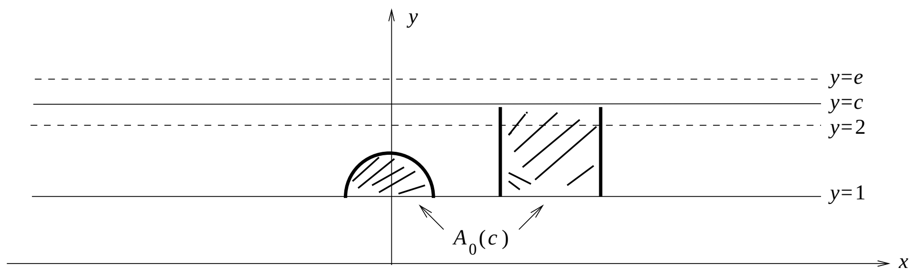

Figure 7: Case $c < e$.

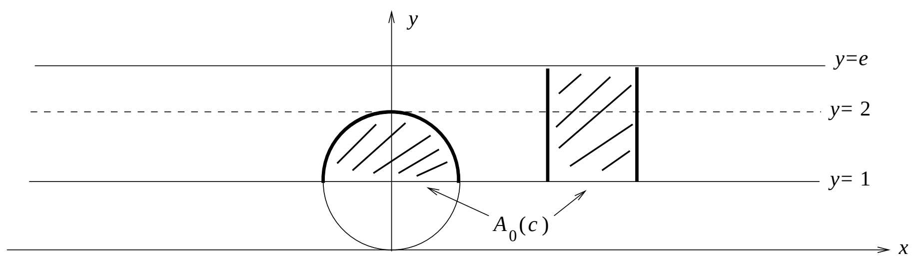

Figure 8: Case $c = e$.

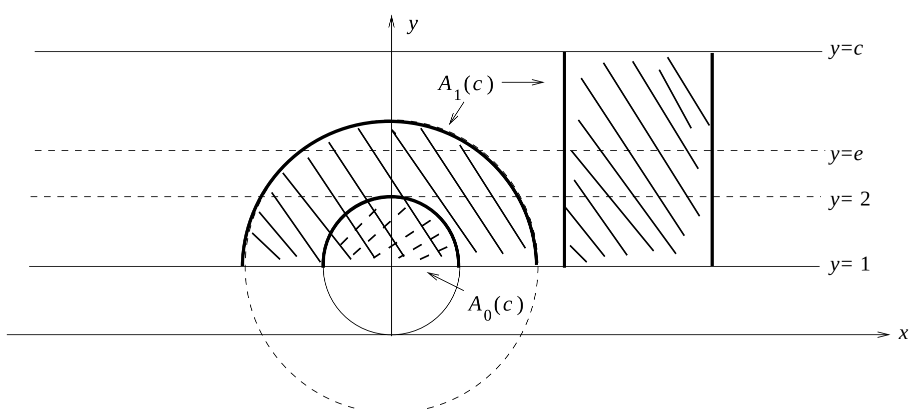

Figure 9: Case $c > e$.